SchrödingerOperator for Sparse Approximation of 3D Meshes

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Abstract

We introduce a Schrödinger operator for spectral approximation of meshes representing surfaces in 3D. The operator is obtained by modifying the Laplacian with a potential function that defines the rate of oscillation of the harmonics on different regions of the surface. We design the potential using a vertex ordering scheme which modulates the Fourier basis of a 3D mesh to focus on crucial regions of the shape having high-frequency structures and employ a sparse approximation framework to maximize compression performance.

Spectral Mesh Compression

- Information of a 3D Mesh comprises of:
  1. **Connectivity: e.g.** mesh triangulation.
  2. **Geometry:** vertex coordinates in $\mathbb{R}^3$ $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^n$. More challenging to compress.
- **Classical Spectral Mesh Compression:** Use spectral geometry of combinatorial Laplacian to encode geometry information:

$$u \approx \sum_{i=1}^{K} \langle u, \phi_i \rangle \phi_i$$

with $K \ll n$ and $\langle u, \phi_i \rangle = \phi_i^T u$.

- **Sparse Representations:** Spectral truncation uses a restrictive assumption by focusing **only** on the first few frequencies.

Sparse Coding: Select representation such that the coefficient vector is *sparse*.

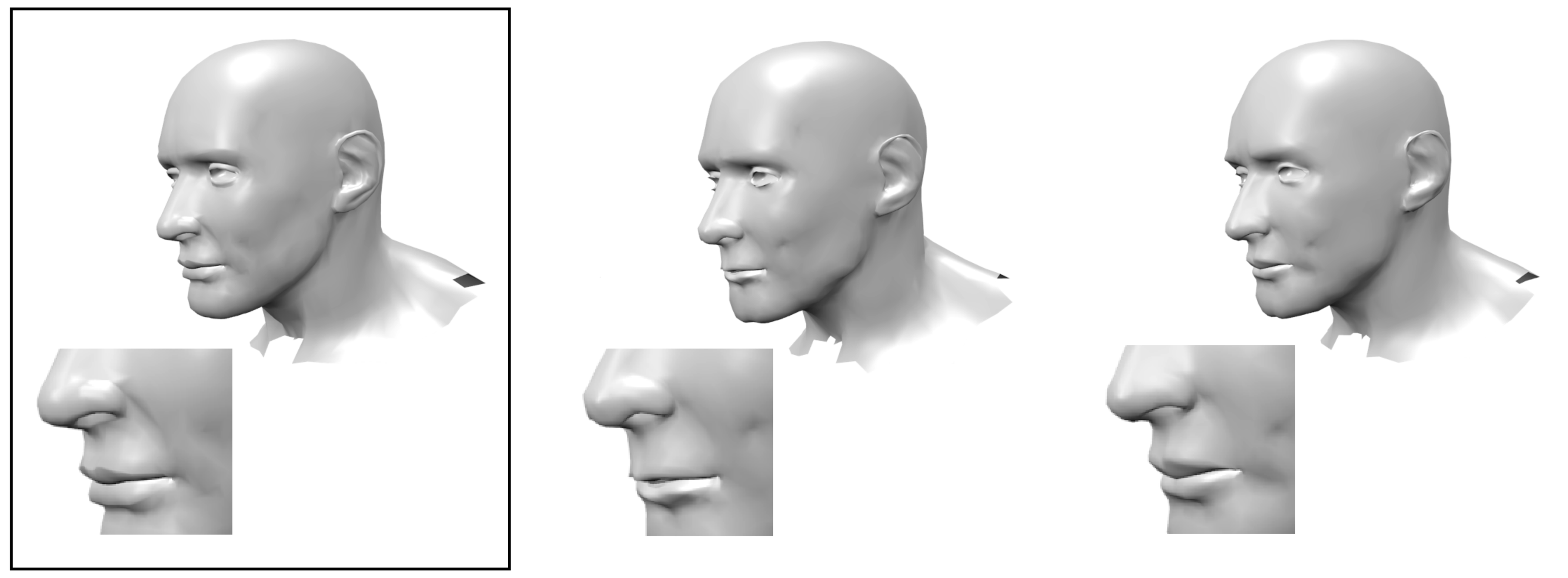
Let $\mathrm{D} \in \mathbb{R}^{n \times m}$ a normalized *overcomplete* dictionary with m atoms $d_i \in \mathbb{R}^n$. u can be obtained as

$$u = D\alpha = \sum_{i=1}^{m} d_i \alpha_i,$$

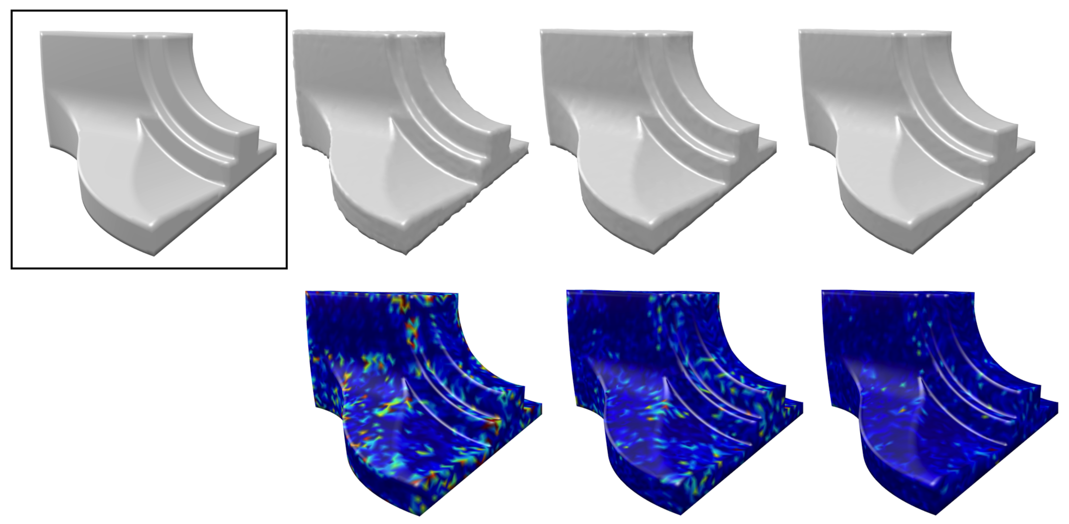
α is obtained from

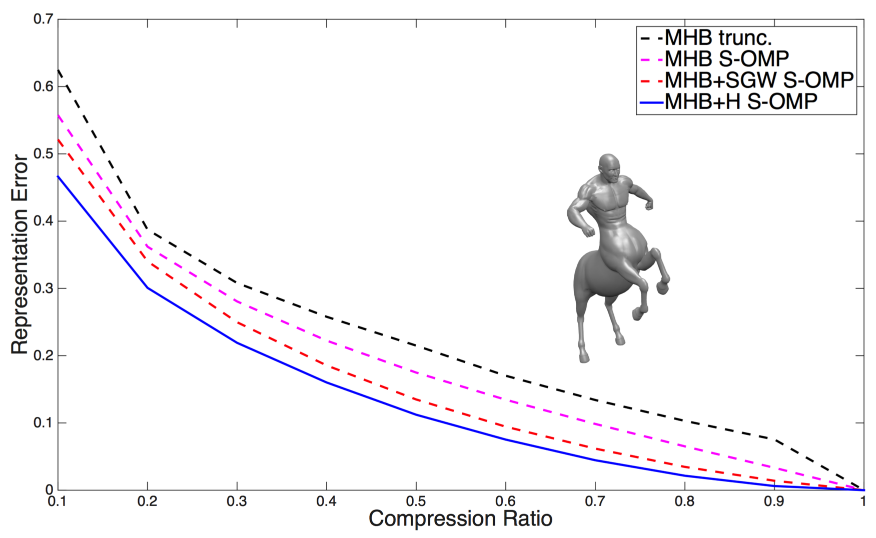
$$\min_{\alpha} \|u - D\alpha\|_2^2 \quad \text{s.t.} \quad \|\alpha\|_0 = k.$$

Results



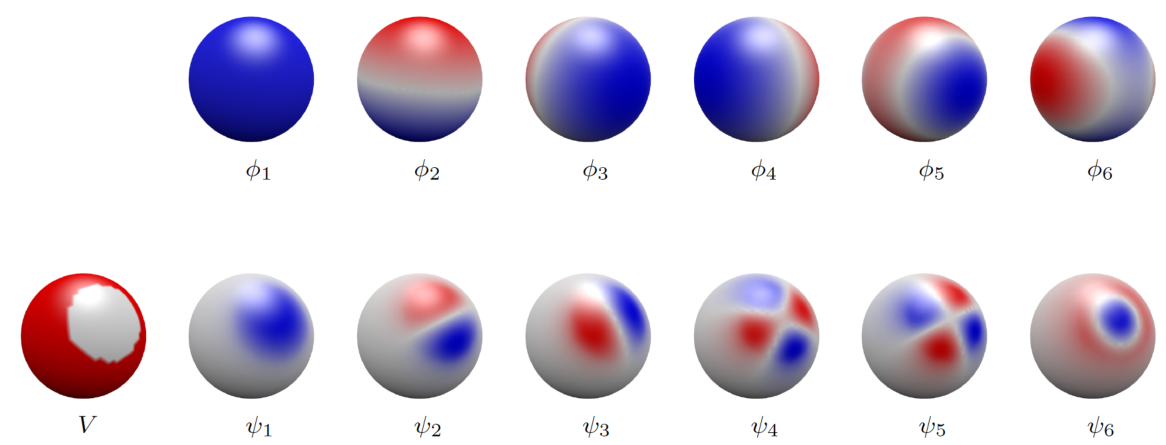
Original (boxed), Laplacian (center) and Hamiltonian (right). Compression ratio = 1:10.





(left) Shape reconstructions for the Fandisk model (12946 vertices). Comparison with Manifold harmonics, Spectral Graph Wavelets and Hamiltonian basis respectively. Compression ratio = 6:10.
(right) Error vs Compression Ratio for Centaur model (15768 vertices).

The Hamiltonian Operator



First eigenfunctions of the LBO on a sphere (top). Potential function defined on the sphere and the corresponding Hamiltonian basis (bottom)

- The Hamiltonian Operator acting on a function f over a manifold $\mathcal{M}$ is given by:

$$H_{\mathcal{M}}(f) = [-\Delta_{\mathcal{M}} + V]\,(f)$$

$\Delta_{\mathcal{M}}$: The Laplace Beltrami Operator on $\mathcal{M}$
$V(x) : \mathcal{M} \to \mathbb{R}$, called the *potential function*.
Discrete eigendecomposition:

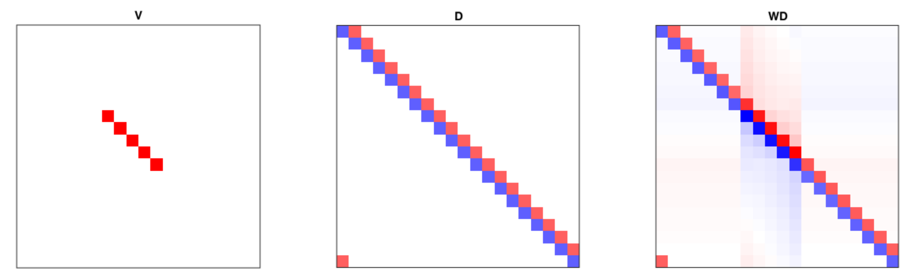
$$\mathrm{H}\psi_i = (\mathrm{L} + \mu \mathrm{V})\psi_i = \mathrm{E}_i \psi_i$$

with $\{\psi_i\}_{i=1}^n \in \mathbb{R}^n$. L: graph Laplacian, and V: potential function at vertices $\pi_i \in \Pi$.

- The Hamiltonian basis can be obtained from the Euler-Lagrange equation of

$$\min_{\psi_i} \int_{\mathcal{M}} \left(\|\nabla_{\mathcal{M}} \psi_i\|_g^2 + \mu V \psi_i^2\right) da, \qquad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle_{\mathcal{M}} = \delta_{ij}.$$

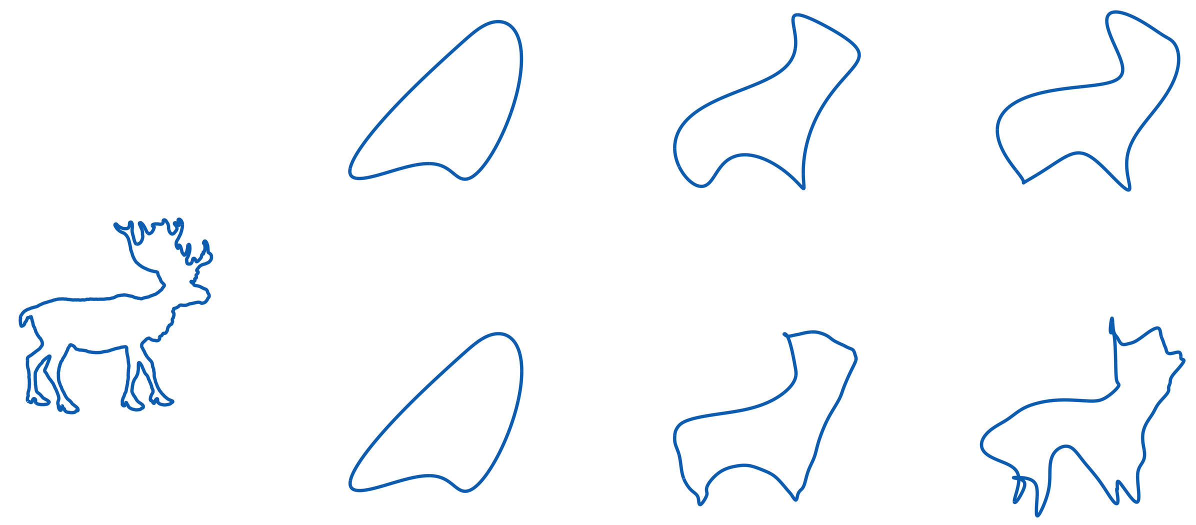
- **1D visualization**: $L = D^T D$ so
$\mathrm{H} = (\mathrm{W}D)^T \mathrm{W}D$ with $\mathrm{W} = (\mathrm{I} + D^{-T}\mathrm{V}D^{-1})^{\frac{1}{2}}$



Matrix visualizations of the potential function (left), the standard gradient matrix D (middle) and the matrix induced from the Hamiltonian (right).

Our Approach

- Design the potential function emphasizing *difficult* regions using the approximation error of the manifold harmonics. The basis optimally adjusts its structure to focus on error-prone regions.
- Use a lossless ordering of the simple vertex in accordance to their approximation error to avoid additional encoding.
- Design dictionary using a sequence of optimized Hamiltonians: $D_j = [\Phi, \Psi_{\mu_1} ..., \Psi_{\mu_j}]$
- Use Simultaneous Orthogonal Matching Pursuit to approximate the co-ordinate functions using the atoms of dictionary D.



Toy example depicting the reconstruction of 2D coordinates using truncation of eigenvectors of the Laplacian (top) and the Hamiltonian (bottom).

References

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