Eurographics Symposium on Geometry Processing (2017)
J. A. Bærentzen and K. Hildebrandt (Editors) *Poster*

Schrödinger Operator for Sparse Approximation of 3D Meshes

Y. Choukroun G. Pai and R. Kimmel

Computer Science Department, Technion – Israel Institute of Technology

Abstract

We introduce a Schrödinger operator for spectral approximation of meshes representing surfaces in 3D. The operator is obtained by modifying the Laplacian with a potential function which defines the rate of oscillation of the harmonics on different regions of the surface. We design the potential using a vertex ordering scheme which modulates the Fourier basis of a 3D mesh to focus on crucial regions of the shape having high-frequency structures and employ a sparse approximation framework to maximize compression performance. The combination of the spectral geometry of the Hamiltonian in conjunction with a sparse approximation approach outperforms existing spectral compression schemes.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations

1. Introduction

In most areas that involve representation of discrete virtual surfaces as 3D meshes, there has been an increasing trend in working with higher precision. This has lead to the generation of meshes which comprise of a large number of elements, for which the processing, visualization and storage of has become a challenge. The task of transmission of these geometric models over communication networks can lead to a large amount of storage space and put a considerable strain on network resources. The information contained in a mesh can be generally divided into two categories: the *geometry* information, which is the position of each vertex of the mesh in the 3D Euclidean space, and the *connectivity* or *topological* information, which describes the incidence relations between the mesh vertices. Since the geometric information comprises a dominant part of the mesh, most recent algorithms focused on its efficient compression.

2. Motivation

The discrete Laplace operator is ubiquitous in spectral shape analysis, since its eigenfunctions are provably optimal in representing smooth functions defined on the surface of the shape [ABK15]. Indeed, subspaces defined by its eigenfunctions, also referred to as manifold harmonics, have been utilized for shape compression, treating the coordinates as approximately smooth functions defined on a given surface. Karni and Gotsman [KG00] were one of the firsts to propose a generalization of the Fourier basis on discrete graphs in order to compress mesh vertex positions. They achieved mesh compression by projecting the coordinate vectors onto the orthonormal basis obtained from the spectral decomposition of the combinatorial Laplacian of the shape. However, surfaces of shapes in nature often contain sharp geometric structures for which the general smoothness assumption, captured by the Laplacian eigenstructure, may fail to hold. Therefore, it is desirable to have a basis which has a larger capacity to encapsulate high-frequency structure. The methods enumerated in [SCOIT05] and [Mah07] provide some perspective on the results in this direction.

Given a basis, a plain spectral truncation of the signal in that basis is a fairly simplistic method for representation which uses a restrictive assumption of only focusing on the lower frequencies of the signal. However, with regards to 3D meshes, local geometry and fine details of the mesh corresponding to high frequencies are generally missing and require a much larger support for their preservation if a truncation approach is followed. Instead, the concept of sparsity and redundant representations provides and alternative perspective for representation where the basic idea is to estimate a given signal as a linear combination of just a few elements (*sparse*) extracted from a large pool of constituent vectors - called a dictionary. These vectors, or atoms, are selected such that the coefficients of representation are sparse. The main difficulty with sparse algorithms is the availability of a rich representative dictionary. This seems trivial for signals defined over regularly and consistently sampled domains, like images and speech, but it is not straightforward to extend the idea to non-flat domains like meshes of surfaces in 3D or general graphs. The use of redundant representations for mesh representation and compression have started to emerge in [TFV06, ZQ14]. Here, we address both these aspects of shape representation. We design a data-aware operator whose spectral geometry is modified by a potential function, in order to emphasize designated regions of interest. We then employ a sparse approximation algorithm which enables efficient information encapsulation in the coefficients for compression.

3. Hamiltonian Operator

A Hamiltonian operator H, also called Schrödinger operator, is an operator acting on a scalar function $f \in L^2(\mathcal{M})$ on a manifold $\mathcal{M}$ that has the form

$$Hf = -\Delta_{\mathcal{M}} f + \mu V f, \tag{1}$$

where $\Delta_{\mathcal{M}}$ is the Laplace Beltrami operator of the surface, $V : \mathcal{M} \to \mathbb{R}$ is called a potential function and $\mu \in \mathbb{R}$. Since the Hamiltonian is a symmetric operator, its eigenfunctions form a complete orthonormal basis on the manifold $\mathcal{M}$. An illustration of the Hamiltonian basis is given in figure 1. The parameter μ defines the trade-off between compactness and global support of the basis. Larger values of μ will give solutions that concentrate on the low potential regions, while a smaller μ will give solutions that better minimize the total energy obtaining smoother wave functions. [HSvTP12, IK12] have used similar approaches for different applications. The addition of the potential function to the Laplacian advocates measuring smoothness differently for different regions of the surface which is a useful property that can be exploited in a representation problem like compression. A simple 1D Euclidean analysis of the differential operator corresponding to the Hamiltonian, shows that the potential modifies the shift-invariance property of the gradient and the effect of a derivative is no more an exclusive property of its local neighborhood but also depends on its global positioning imposed by the potential function.

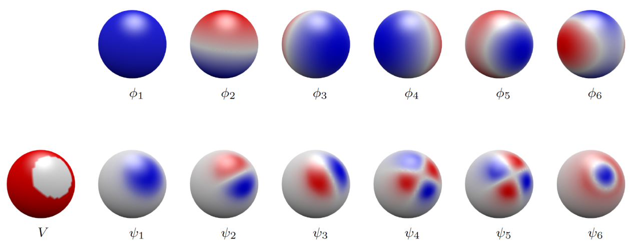

Figure 1: *First eigenfunctions of the LBO on a sphere (top). Potential function defined on the sphere and the corresponding Hamiltonian basis (bottom). Hot and cold colors represent positive and negative values respectively. The Hamiltonian basis concentrates the harmonics to the low potential region.*

4. Discussion

Our method is designed to take advantage of the two principle components of our contribution: sparsity and data-dependent basis. Our main observation is that the potential function in Equation 1 can be *designed* so that the manifold harmonics are altered, in order to focus on the high-frequency structures and finer details of the 3D mesh. In order to achieve this, we choose the potential to be a function of the approximation error of the mesh coordinates using the standard Manifold harmonics. By doing simple pre-computation of ordering the vertices in accordance with this error and designing a fixed structure potential, we avoid the need to encode this additional information. Sorting the vertices in order of their approximation error highlights crucial regions of the shape having finer high-frequency geometric structures and therefore difficult to compress for the regular Laplacian. Thus, the Hamiltonian operator can improve the compression performance by modifying the harmonics in order to emphasize *designated* regions of interest.

We construct a dictionary built from the eigendecomposition of both the Laplacian and the Hamiltonian to benefit from their global and local properties. Atoms are selected by using a simultaneous pursuit approach. We encode multiple constants μ in order to obtain a multiresolution of the basis. These regularization constants are found via a direct search on a given domain. Thus, by designing high vibrations in selected regions of the shape, our dictionary is much less redundant than the wavelets proposed in [Mah07], has a better ability to encode localized high frequency details and achieves better compression performance.

We evaluated our results using the metric proposed in [KG00] which is a linear combination of the RMS geometric distance between corresponding vertices in both models and a visual metric capturing the smoothness of the reconstruction. In order to ease the demand of a computationally intensive eigendecomposition of a large matrix and the sparse coding, we resort to mesh partitioning, where we segment a mesh into smaller constituents and compress each segment independently. The compression ratio calculation is obtained as a ratio of the net information before and after the dictionary encoding [ZQ14] where both the atoms and their coefficients are encoded. Our results compare favorably with existing spectral approaches using Manifold Harmonics and Spectral Graph Wavelets using sparse approximation approaches or not. The main advantage gained is due to the shape-adaptability of our basis.

The authors would like to acknowledge that they are supported by Grant agreement No. 267414 of the European Communities FP7-ERC program.

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